Curves and surfaces $\boxed{\text{Week 5}}$

The fundamental theorem of curves in $\mathbb{R}^3$:

Thm: Let $\kappa, \tau : I \to \mathbb{R}$ be continuous functions with $\kappa > 0$. There exists a Frenet regular and naturally parametrized curve $\gamma : I \to \mathbb{R}^3$ such that

$$\kappa_\gamma(s) = \kappa(s), \quad \tau_\gamma(s) = \tau(s).$$

This curve is unique up to a rigid motion (i.e. an isometry of $\mathbb{R}^3$) preserving the orientation

Proof:

As we mentioned last time, if $\gamma : I \to \mathbb{R}^3$ is Frenet regular, the matrix

$$F_\gamma(s) = [T_\gamma(s) \mid N_\gamma(s) \mid B_\gamma(s)]$$

satisfies:

- $F_\gamma \in SO(3)$, since $\{T_\gamma, N_\gamma, B_\gamma\}$ is a positively oriented orthonormal basis
- $F_\gamma(s)$ is the transformation matrix for the change of basis $\{e_1, e_2, e_3\} \longrightarrow \{T_\gamma, N_\gamma, B_\gamma\}$
- The Serret-Frenet formulas take the form:

$$\frac{d}{ds} F_\gamma(s) = F_\gamma(s) \cdot \Omega_\gamma(s) \quad ①, \quad \text{where } \Omega_\gamma(s) = \begin{pmatrix} 0 & -\kappa_\gamma(s) & 0 \\ +\kappa_\gamma(s) & 0 & -\tau_\gamma(s) \\ 0 & +\tau_\gamma(s) & 0 \end{pmatrix}$$

Proof of uniqueness: Suppose that we have two curves γ_1, γ_2 solving ① with

$$\Omega(s) = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix}.$$

By composing each curve with an isometry of $\mathbb{R}^3$ (i.e. $T_1 \circ \gamma_1$, $T_2 \circ \gamma_2$) we can assume without loss of generality that (assume also $0 \in I$)

1) $\gamma_1(0) = \gamma_2(0) = 0$ 2)

2) $\{T_{\gamma_1}(0), N_{\gamma_1}(0), B_{\gamma_1}(0)\} = \{T_{\gamma_2}(0), N_{\gamma_2}(0), B_{\gamma_2}(0)\} = \{e_1, e_2, e_3\}$

(so that $F_1(0) = F_2(0) = \mathbb{I}$).

Then both $F_1(s)$, $F_2(s)$, solve the same equation ① with the same initial data. We will show that this implies that $F_1(s) = F_2(s)$ $\forall s \in I$.

$$\frac{d}{ds}(F_1 \cdot F_2^{-1}) \overset{F_2 \in SO(3)}{=} \frac{d}{ds}(F_1 \cdot F_2^T) = \dot{F}_1 F_2^T + F_1 \cdot \dot{F}_2^T$$

$$= F_1 \Omega F_2^T + F_1 \Omega^T F_2^T = 0, \quad \text{since } \Omega = -\Omega^T.$$

Since $F_1 \cdot F_2^{-1}\big|_{s=0} = \mathbb{I} \Rightarrow F_1(s) F_2^{-1}(s) = \mathbb{I}$ $\forall s \in I \Rightarrow F_1(s) = F_2(s)$

In particular, since the corresponding columns are also equal:

$$T_{\gamma_1}(s) = T_{\gamma_2}(s) \quad \forall s \in I \overset{|\dot\gamma_i| = 1}{\iff} \dot{\gamma}_1(s) = \dot{\gamma}_2(s) \quad \forall s$$

Integrate in s: $\gamma_1(s) - \cancelto{0}{\gamma_1(0)} = \gamma_2(s) - \cancelto{0}{\gamma_2(0)}$ $\forall s \in I$

Proof of existence:

Consider the 1^{st} order matrix-valued ODE ①:

$$\begin{cases} \dfrac{d}{ds} F(s) = F(s) \cdot \Omega(s) \\ F(0) = \mathbb{I} \end{cases}$$

Since $\Omega(s) \in C^0$: By the Cauchy-Lipschitz theorem, $\exists$ C^1 solution $F : I \to M_3(\mathbb{R})$. We will show that, in fact, $F(s) \in SO(3)$ $\forall s \in I$.

$$\frac{d}{ds}(F \cdot F^T) = \dot{F} F^T + F \cdot \dot{F}^T = F \Omega F^T + F \Omega^T F^T = 0, \quad \text{since } \Omega = -\Omega^T.$$

So $F(s) \cdot F^T(s) = F(0) \cdot F^T(0) = \mathbb{I} \Rightarrow F(s) \in O(3)$

And: $\det F(s)$ is continuous, so, since $\det F(s) = \pm 1$ and $\det F(0) = +1$, we must have $\det F(s) = +1 \quad \forall s \quad \Rightarrow \quad F(s) \in SO(3)$.

Let's denote by $T(s)$, $N(s)$, $B(s)$ the columns of $F(s)$ (this is an orthonormal, positively oriented basis).

Define the curve γ by

$$\gamma(s) = \int_0^s T(u)\, du.$$

Then: $\dot{\gamma}(s) = T(s)$ (so $\|\dot{\gamma}\| = 1$ and T is the unit tangent vector)

The equation ① implies that $\frac{d}{ds} T(s) = \kappa(s) N(s)$, so

$N(s) = N_\gamma(s)$ (since $N_\gamma \nearrow\!\!\nearrow \dot{T}_\gamma$ and is unit)

and $\kappa(s) = \kappa_\gamma(s)$

Then, $B_\gamma(s) = T_\gamma(s) \times N_\gamma(s) = T(s) \times N(s) \overset{\{T,N,B\}\text{: orthonormal, pos. oriented}}{=\!=} B(s)$

and ① is the Serret-Frenet formulas, so $\tau(s) = \tau_\gamma(s)$ ☒

Curves of constant slope:

Def: A curve of constant slope (or a generalised helix) is a regular curve in $\mathbb{R}^3$ such that T_γ forms a constant angle with a fixed direction $A \in \mathbb{R}^3$.

Prop: Let $\gamma : I \to \mathbb{R}^3$ be Frenet regular. It has constant slope if and only if $\frac{\tau_\gamma(u)}{\kappa_\gamma(u)}$ is constant

Proof: By reparametrizing γ if necessary, assume it is naturally parametrized

- If γ has constant slope, then there exists a <u>unit</u> vector $A \in \mathbb{R}^3$ such that $\langle T_\gamma(s), A \rangle = \cos\theta = \text{const}$

Differentiating: $0 = \langle \dot{T}_\gamma, A \rangle \overset{\text{Serret-Frenet}}{=} \kappa_\gamma \langle N_\gamma, A \rangle$.

Since γ is biregular $\Rightarrow \kappa_\gamma \neq 0 \Rightarrow \langle N_\gamma(s), A \rangle = 0 \ \forall s$.

Since $\langle T_\gamma, A \rangle \underset{= \cos\theta}{=} \text{const}$ and $\langle N_\gamma, A \rangle = 0$

$\Rightarrow \langle B_\gamma, A \rangle = \langle T_\gamma \times N_\gamma, A \rangle = \sin\theta = \text{const}$

From the Serret-Frenet formulas:

$$\dot{N}_\gamma = -\kappa_\gamma T_\gamma + \tau_\gamma B_\gamma \Rightarrow \underset{\substack{\| \\ 0}}{\frac{d}{dt}\langle N_\gamma, A \rangle} = -\kappa_\gamma \langle T_\gamma, A \rangle + \tau_\gamma \langle B_\gamma, A \rangle$$

$$\Rightarrow \frac{\tau_\gamma}{\kappa_\gamma} = \frac{\langle T_\gamma, A \rangle}{\langle B_\gamma, A \rangle} = \text{const} = \cot\theta$$

- For the converse: If $\frac{\tau_\gamma}{\kappa_\gamma} = \text{const} = a$, then set

$A(s) = a\, T_\gamma(s) + B_\gamma(s)$. Then

$\dot{A}(s) = a \cdot \dot{T}_\gamma + \dot{B}_\gamma = a \cdot \kappa_\gamma \cdot N_\gamma - \tau_\gamma N_\gamma = 0$ so

A is constant and $\langle A, T_\gamma \rangle = a = \text{const}$ $\square$

<u>Curves in the oriented plane</u>

In what follows, we will assume that we have fixed an orientation of $\mathbb{R}^2$. We can adapt the Frenet formalism to planar curves.

<u>Def:</u> Let $\gamma: I \to \mathbb{R}^2$ be <u>regular</u> of class C^1. The <u>(oriented) Frenet</u>

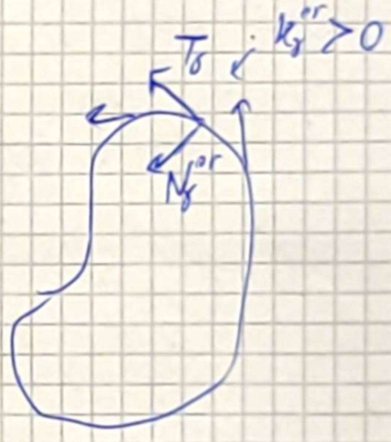
frame of γ is the pair of orthonormal vector fields:

$$T_\gamma(u) = \frac{1}{V_\gamma(u)}\dot\gamma(u), \qquad N^{or}_\gamma(u) = J(T_\gamma(u)).$$

b) If, in addition, γ is of class C^2, oriented curvature

$$\kappa^{or}_\gamma(u) = \frac{1}{V_\gamma(u)} \langle \dot T_\gamma(u), N^{or}_\gamma(u)\rangle.$$

Remarks:



1) $\{T_\gamma, N^{or}_\gamma\}$: positively oriented, orthonormal.

2) κ^{or}_γ can take any real values,

$$|\kappa^{or}_\gamma| = \kappa_\gamma.$$

In fact $\kappa_\gamma(u) = \frac{1}{V_\gamma(u)} T_\gamma(u) \wedge \dot T_\gamma(u)$

3) If the curve is biregular:

$$\kappa^{or}_\gamma(u) \cdot N^{or}_\gamma(u) = \kappa_\gamma(u) N_\gamma(u) = K_\gamma(u)$$

4) If we change the orientation of the plane: κ^{or}_γ, N^{or}_γ change sign. (same if we reparametrize γ by an inversion)

The Serret-Frenet formulas in 2d:

For a regular curve of class C^2:

$$\frac{1}{V_\gamma}\frac{d}{dt} T_\gamma = \kappa^{or}_\gamma N^{or}_\gamma$$
$$\frac{1}{V_\gamma}\frac{d}{dt} N_\gamma = -\kappa^{or}_\gamma T_\gamma$$

or

$$\frac{1}{V_\gamma}\frac{d}{dt}\begin{pmatrix} T_\gamma \\ N_\gamma \end{pmatrix} = \begin{pmatrix} 0 & \kappa^{or}_\gamma \\ -\kappa^{or}_\gamma & 0 \end{pmatrix}\begin{pmatrix} T_\gamma \\ N_\gamma \end{pmatrix}$$

Proof: Since $\langle T_\gamma, T_\gamma\rangle = 1$, $\langle T_\gamma, N^{or}_\gamma\rangle = 0$, $\langle N^{or}_\gamma, N^{or}_\gamma\rangle = 1$:

Differentiating the above: $\langle \dot T_\gamma, T_\gamma\rangle = 0$, $\langle \dot T_\gamma, N^{or}_\gamma\rangle = -\langle T_\gamma, \dot N^{or}_\gamma\rangle$,

$$\langle \dot N^{or}_\gamma, N^{or}_\gamma\rangle = 0$$

So if $$\frac{1}{V_\gamma}\dot{T}_\gamma = a\,T_\gamma + b\,N_\gamma^{or}$$
$$\frac{1}{V_\gamma}\dot{N}_\gamma^{or} = c\,T_\gamma + d\,N_\gamma^{or}$$
then $a = \frac{1}{V_\gamma}\langle \dot{T}_\gamma, T_\gamma\rangle = 0$, $b = \frac{1}{V_\gamma}\langle \dot{T}_\gamma, N_\gamma^{or}\rangle = \kappa_\gamma^{or}$ (definition of κ_γ^{or}),

$c = \frac{1}{V_\gamma}\langle \dot{N}_\gamma^{or}, T_\gamma\rangle = -\frac{1}{V_\gamma}\langle N_\gamma^{or}, \dot{T}_\gamma\rangle = -\kappa_\gamma$, $d = \frac{1}{V_\gamma}\langle \dot{N}_\gamma^{or}, N_\gamma\rangle = 0$ ☐

<u>Lemma</u>: If $\gamma: I \to \mathbb{R}^2$ is regular of class C^2:
$$\kappa_\gamma^{or}(u) = \frac{\dot{\gamma}(u)\wedge\ddot{\gamma}(u)}{V_\gamma^3(u)}.$$

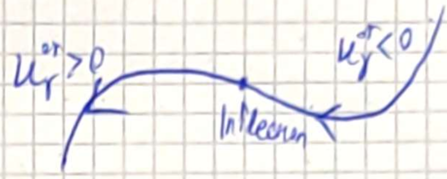
<u>Proof</u>: $$\dot{\gamma}\wedge\ddot{\gamma} = V_\gamma T_\gamma \wedge (V_\gamma^2 \underbrace{\kappa_\gamma N_\gamma}_{=\kappa_\gamma^{or} N_\gamma^{or}} + \dot{V}_\gamma T_\gamma)$$
$$= V_\gamma^3\, T_\gamma \wedge (\kappa_\gamma^{or} N_\gamma^{or})$$
$$= V_\gamma^3\, \kappa_\gamma^{or}.$$ ☐

So: If $\gamma(t) = (x(t), y(t))$,
$$\kappa_\gamma^{or}(t) = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

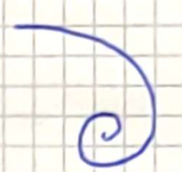
If $\gamma(x) = (x, f(x))$:
$$\kappa_\gamma^{or}(x) = \frac{f''}{(1+(f')^2)^{3/2}}$$

Definitions:

- Convex arc: $\kappa_\gamma^{or} > 0$
- Concave arc: $\kappa_\gamma^{or} < 0$
- Inflection point: Point where κ_γ^{or} changes sign



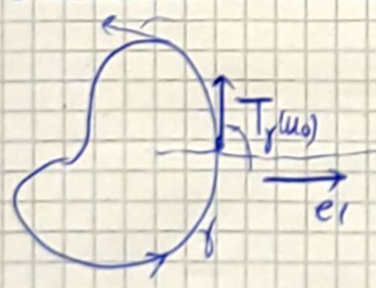
- Spiral: If κ_γ^{or} strictly monotonic



- Vertex: a point where $\dot{\kappa}_\gamma^{or} = 0$

At a vertex: The osculating circle has contact of order $\geqslant 3$ with γ (while, when $\dot{\kappa}_\gamma^{or} \neq 0$, they have contact of order 2).
The vertex corresponds to a cusp point of the involute

The angular function:



Def: Let $\gamma: I \to \mathbb{R}^2$ be regular.
Angular function with initial point $\gamma(u_0)$:
$\phi: I \to \mathbb{R}$ such that

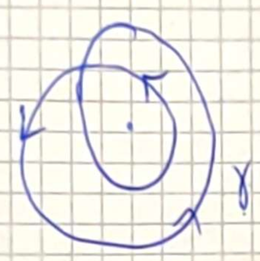
i) $\phi(u_0)$ = oriented angle between $T_\gamma(u_0)$ and e_1
ii) ϕ is continuous
iii) Oriented angle between $T_\gamma(u)$, e_1 is $\phi(u)$ mod 2π.

Note: Every full rotation of $T_\gamma(u)$ as I move along γ increases $\phi(u)$ by 2π.

If $I = [a,b]$: Total angular variation $\phi(b) - \phi(a)$.

If γ is periodic $(\gamma(a) = \gamma(b),\ \dot\gamma(a) = \dot\gamma(b))$

Then $\dfrac{\phi(b) - \phi(a)}{2\pi}$ is an integer. (number of rotations of γ around a point in the interior)
(winding number)



<u>Lemma</u>: $T_\gamma = (\cos\phi, \sin\phi)$, $N^{or}_\gamma = (-\sin\phi, \cos\phi)$

<u>Proof</u>: T_γ is by definition of ϕ like this
(ϕ mod 2π is the angle between T_γ and e_1).
$N^{or}_\gamma = J(T_\gamma)$.

<u>Lemma</u>: If γ is of class C^2,

$$\kappa^{or}_\gamma(t) = \frac{1}{V_\gamma}\frac{d}{dt}\phi(t).$$

<u>Proof</u>: $\kappa^{or}_\gamma = \frac{1}{V_\gamma}\langle \dot T_\gamma, N^{or}_\gamma \rangle = \frac{1}{V_\gamma} \cdot \dot\phi \cdot \langle (-\sin\phi, \cos\phi), (-\sin\phi, \cos\phi) \rangle.$

With respect to natural parametrization:

$$\kappa^{or}_\gamma(s) = \frac{d\phi}{ds}$$

So when using ϕ as a parameter (sometimes convenient):

$$d\phi = \kappa^{or}_\gamma\, ds.$$

By integrating: If γ is <u>periodic</u> and <u>simple</u> (no self intersection)

then $\boxed{\int \kappa^{or}_\gamma\, ds = \pm 2\pi}$

and, for any periodic curve

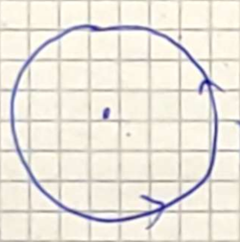
$$\underbrace{\int_a^b \kappa_\gamma \, ds}_{|\kappa_\gamma^{or}|} \geq 2\pi.$$

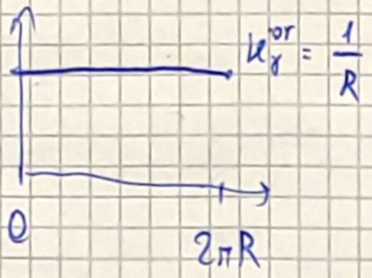
Note: Same is true for closed space curves $\gamma: I \to \mathbb{R}^3$.

Fáry-Milnor: If $\gamma: I \to \mathbb{R}^3$ is knotted, then $\int_I \kappa_\gamma \, ds \geq 4\pi$.

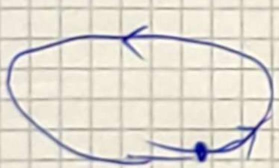
The fundamental theorem of curves in the plane:

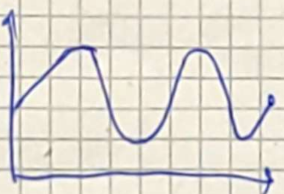
Curvature diagram: The graph of $\kappa_\gamma^{or} = \kappa_\gamma^{or}(s)$ as a function of the natural parameter





We can read off vertices (local extrema), inflection points, etc.





$\int_I \kappa_\gamma^{or} \, ds$ = total angular variation.

The diagram determines γ up to a rigid motion of $\mathbb{R}^2$:

Theorem: For any continuous $\kappa: [0,l] \to \mathbb{R}$, there exists a naturally parametrised curve with $\kappa_\gamma^{or}(s) = \kappa(s)$; this curve is unique up to rigid motion of $\mathbb{R}^2$.

Proof: Same (but simpler) as in the 3d case. Instead of the matrix ODE for $F_\gamma(s) = [T_\gamma(s) \vdots N_\gamma^{or}(s)]$, one can consider the ODE for $\phi(s)$: $\phi'(s) = \kappa(s)$, $\phi(s)$ has all the information contained in $F_\gamma(s) = \begin{pmatrix}\cos\phi & -\sin\phi \\ \sin\phi & \cos\phi\end{pmatrix}$.

Since $\kappa(s)$ determines $\phi(s)$ (assuming $\phi(0)=\phi_0$):

One recovers $\gamma(s)$ by the relation

$$\gamma(s) = \gamma(0) + \int_0^s \dot{\gamma}(\tau)\, d\tau = \gamma(0) + \int_0^s \left(\cos(\phi(\tau)), \sin(\phi(\tau))\right) d\tau.$$

The 4-vertices theorem

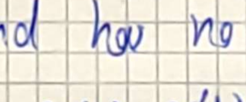
Def: A curve $\gamma : [a,b] \to \mathbb{R}^n$ is a closed curve of class C^m if it is of class C^m and

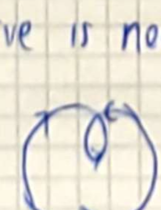
$$\gamma(a) = \gamma(b) \quad \text{and} \quad \frac{d^k\gamma}{dt^k}(a) = \frac{d^k\gamma}{dt^k}(b) \quad \text{for } 1 \leq k \leq m$$

It is simple if it is regular and has no double points (i.e. no self intersections), other than $\gamma(a)=\gamma(b)$.

Note: A closed C^m curve can be extended to a periodic C^m curve $\gamma : \mathbb{R} \to \mathbb{R}^n$

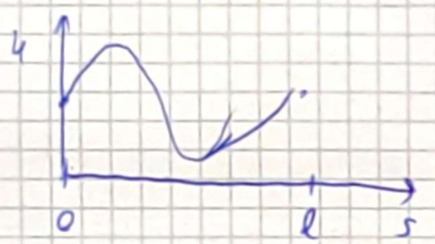
Theorem: Let $\gamma : [a,b] \to \mathbb{R}^3$ be a simple, closed curve of class C^3. Then γ has at least 4 vertices

Remarks:
- Ellipse: Exactly 4 
- Circle: Every point is a vertex
- If the curve is not simple: Not true

e.g.  has 2 vertices.

Proof: Assume (w.l.o.g) that γ is parametrized by arc length. Since it is C^3 and closed:

$\kappa(0) = \kappa(\ell)$, $\dot{\kappa}(0) = \dot{\kappa}(\ell)$ (here: $\kappa = \kappa_\gamma^{or}$)



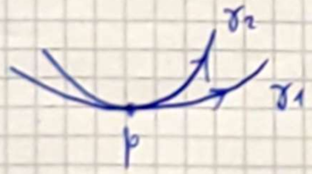
$\leftarrow$ So κ: has at least two critical points (at least a local minimum and a local maximum).

If it has 3 critical points: It should necessarily have a fourth one as well (e.g, between two local minima there is always a local maximum).

We will follow the proof by Osserman (but check also the proof in do Carmo's book).

Lemma 1:

Assume that γ_1 is a circle of radius R parametrized counter clockwise and γ_2 is a curve inside the circle touching γ_1 at p, so that $T_{\gamma_1} = T_{\gamma_2}$ there



Then $\kappa_{\gamma_2}^{or}|_p \geq \frac{1}{R} = \kappa_{\gamma_1}^{or}$

Proof: By a rigid motion, assume that both are graphs of functions f_1, f_2 over x, such that $f_1(0) = f_2(0) = 0$, $f_1'(0) = f_2'(0) = 0$.

Then γ_2 inside $\gamma_1 \Rightarrow f_2 \geq f_1 \Rightarrow f_2''(0) \geq f_1''(0)$ ▨

(proof to be continued next time)